J. Fluid Mech. (1967), *vol.* 27, *part* 4, *pp.* 705–724
Printed in Great Britain

On the slow motion of a sphere parallel to a nearby plane wall

By M. E. O'NEILL AND K. STEWARTSON
Department of Mathematics, University College, London

(Received 7 March 1966)

A new method using a matched asymptotic expansions technique is presented for obtaining the Stokes flow solution for a rigid sphere of radius a moving uniformly in a direction parallel to a fixed infinite plane wall when the minimum clearance ϵa between the sphere and the plane is very much less than a. An 'inner' solution is constructed valid for the region in the neighbourhood of the nearest points of the sphere and the plane where the velocity gradients and pressure are large; in this region the leading term of the asymptotic expansion of the solution satisfies the equations of lubrication theory. A matching 'outer' solution is constructed which is valid in the remainder of the fluid where velocity gradients are moderate but it is possible to assume that $\epsilon = 0$. The forces and couples acting on the sphere and the plane are shown to be of the form $(\alpha_0+\alpha_1\epsilon)\log\epsilon+\beta_0+O(\epsilon)$ where α_0, α_1 and β_0 are constants which have been determined explicitly.

1. Introduction

The problem of the slow motion of two rigid spheres through a viscous fluid has been of interest for many years, because of its importance to the flow of suspensions and the theory of sedimentation. The earliest attack on it was by Smoluchowski (1911) who considered the flow properties when the spheres are far apart by an iteration procedure, in which the spheres were treated separately and alternately. His method was afterwards extended by Faxén (1927), Burgers (1941), Kynch (1959) and others. A more satisfactory approach in the special case of two spheres in motion along their common diameter only, so that the flow is axi-symmetrical, was developed by Stimson & Jeffrey (1926) using bipolar co-ordinates. They were able to obtain the solution as a series, which converges rapidly except when the spheres are almost in contact. An account of this work has been given by Happel & Brenner (1965). Recently Dean & O'Neill (1963) and O'Neill (1964*a*, *b*) have extended the use of bipolar co-ordinates to asymmetrical problems such as rotating spheres and motion at right angles to their common diameter. Again the solution is expressed as a series, but now the coefficients of the various terms cannot be determined except as the solution of a set of difference equations. However, these equations can easily be programmed for a high-speed computer. O'Neill (1964*a*) has given the principal properties of the flow in the special case when one of the spheres degenerates into a plane in an accurate tabular form, even when the distance between the sphere and the plane is small.

Other cases have been studied by Majumdar (1965). In principle therefore it may now be claimed that the problem of the two spheres moving slowly through a viscous medium is solved, apart from certain limiting situations when the series obtained either fail to converge or converge too slowly for numerical convenience. The aim of the present paper is to complement the work of Dean & O'Neill by describing a method especially suitable for these limiting cases and to apply it to perhaps the most important of them, when the sphere is very close to a fixed plane ($\epsilon \ll 1$) and moving parallel to it as shown in figure 1.

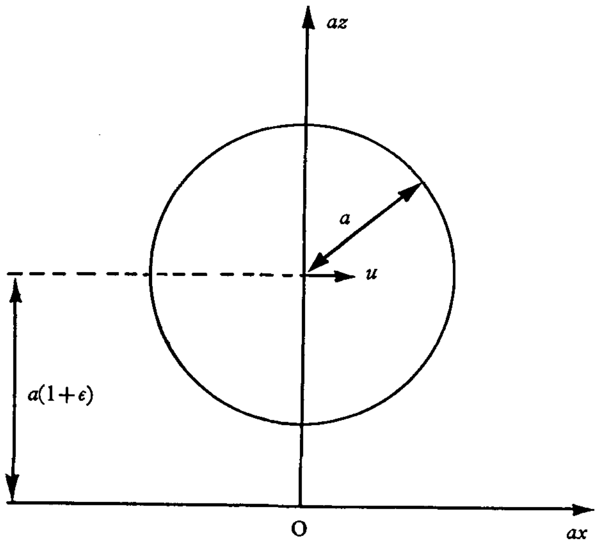

FIGURE 1.

This flow problem is also of interest in another connexion. The theory of lubrication is extremely important for engineering design, and although it has been extensively developed for many years (see Pinkus & Sternlicht 1961 and Gross 1962 for an account of the present state of the theory) it suffers from a number of drawbacks.

First, it is difficult to perform experiments which would adequately test the mathematical predictions, partly because of problems of design and partly because there are no lubricants whose viscosity is independent of temperature, pressure and rate of shear. Secondly, the theory has developed on an *ad hoc* basis; a number of assumptions being made without proper discussion of their range of validity and systematic attempts to embed the theory in a rational approximation scheme for the solution of the Navier–Stokes equations are only now beginning to be undertaken (Langlois 1964; Thompson 1964). Consequently it is not easy to make *quantitative* assessments of the effects of such phenomena as the compressibility, inertia and non-constant viscosity of the fluid and the non-rigidity of the boundaries. Furthermore, in certain types of problem, such as the partially filled bearing, what is known about the actual flow properties makes one somewhat sceptical of the *sufficiency* of lubrication theory.

In the face of all these difficulties in the theory, the contribution of this paper is the modest one of assessing its accuracy when applied to a fluid motion in which there is also a large region in which the flow is weakly sheared. Lubrication theory gives a valid first approximation to the flow in the neighbourhood of O in figure 1 provided that the Reynolds number based on minimum clearance between the sphere and plane is small and $\epsilon \ll 1$. Our solution enables us to embed this theory into an exact solution of the Stokes equations for creeping flow, and hence we can make an estimate of the accuracy of the theory with regard to local and overall properties of the flow.

The method of solution we adopt is to divide the flow into two parts. First, there is an inner region, in the neighbourhood of O in which the velocity gradients and the pressure are large. In this region the leading term of the asymptotic expansion of the solution of the Stokes equations satisfies the equations of lubrication theory and successive terms can be obtained in a straight-forward way. Secondly, there is an outer region consisting of the remainder of fluid in which velocity gradients are moderate but it is possible to assume that $\epsilon = 0$. The co-ordinate system may then be inverted about O reducing the problem to the flow between two parallel planes with singularities at infinity (i.e. at the inner edge of the outer region, near O). These singularities lead to a perfect match with the structure of the inner solution leaving the neighbourhood of O. In this way a completely consistent solution is obtained when $\epsilon \ll 1$ which may, if desired, be made the first term in an asymptotic solution of the Stokes equations.

2. The statement of the problem

It is supposed that the fluid motion is generated by the sphere, which is rigid and has radius a, moving with uniform velocity $(\mathscr{U}, 0, 0)$ referred to a system of Cartesian co-ordinates (ax, ay, az) in which the plane is $z = 0$ and, instantaneously, the co-ordinates of its centre are $(0, 0, a(1+\epsilon))$. Then the minimum clearance between the sphere and the plane is ϵa. It is further supposed that the fluid is incompressible, has a constant density ρ and viscosity μ, and that the Reynolds number $\mathscr{U}a\rho/\mu$ is sufficiently small to permit the neglect of the inertia terms in the Navier–Stokes equations. The equations governing the fluid motion are therefore

$$\nabla p = \mu\nabla^2\mathbf{V}, \quad \operatorname{div}\mathbf{V} = 0, \tag{1}$$

where p is the pressure and $\mathbf{V}$ the velocity. The boundary conditions are that $\mathbf{V} \to 0$ at infinity since the fluid is supposed at rest there, and $\mathbf{V} = 0$ on the plane. Further at any point of the sphere the components (u, v, w) of $\mathbf{V}$ in cylindrical polar co-ordinates (ar, θ, az), in which the origin is that point of plane nearest the sphere, are given by

$$u = \mathscr{U}\cos\theta, \quad v = -\mathscr{U}\sin\theta, \quad w = 0. \tag{2}$$

The method of solution of this problem is to divide the flow region into an inner region in which r, z are both small and flow properties are similar to those in lubrication problems, and an outer region comprising the rest of the flow in which essentially ϵ may be set equal to zero and the flow properties are obtained by means of Fourier–Bessel transforms. We first consider the inner region.

3. The inner region

Writing $ap = \mu\mathscr{U}P\cos\theta$ and the cylindrical components of $\mathbf{V}$ as $(U\cos\theta,$ $V\sin\theta,\ W\cos\theta)$, equations (2) are satisfied if

$$\partial P/\partial r = L_0^2 U - 2(U+V)/r^2, \tag{3}$$

$$-P/r = L_0^2 V - 2(U+V)/r^2, \tag{4}$$

$$\partial P/\partial z = L_1^2 W, \tag{5}$$

and

$$\frac{\partial U}{\partial r} + \frac{U+V}{r} + \frac{\partial W}{\partial z} = 0, \tag{6}$$

the operator L_m^2 being defined by

$$L_m^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2}. \tag{7}$$

The clearance $a\delta$ between the sphere and the plane, expressed as a function of r, is given by

$$\delta = 1 + \epsilon - (1-r^2)^{\frac{1}{2}}. \tag{8}$$

Now δ and therefore z are $O(\epsilon)$, which together with (8) suggests the introduction of new variables, R, Z, defined by

$$r = \epsilon^{\frac{1}{2}}R, \quad z = \epsilon Z.$$

Thus (8) now gives

$$\delta = \epsilon H + \tfrac{1}{8}\epsilon^2 R^4 + \ldots, \tag{9}$$

where $H = 1 + \frac{1}{2}R^2$.

The boundary conditions imply that U and V are $O(1)$. Consequently (6) implies that $W = O(\epsilon^{\frac{1}{2}})$ and (4) implies that $P = O(\epsilon^{-\frac{3}{2}})$, which suggests that we look for solutions for P, U, V and W of the form

$$\left.\begin{aligned} P(r,z) &= \epsilon^{-\frac{3}{2}}\,P_0(R,Z) + \epsilon^{-\frac{1}{2}}P_1(R,Z) + \ldots, \\ U(r,z) &= \quad U_0(R,Z) + \epsilon U_1(R,Z) + \ldots, \\ V(r,z) &= \quad V_0(R,Z) + \epsilon V_1(R,Z) + \ldots, \\ W(r,z) &= \epsilon^{\frac{1}{2}}\,W_0(R,Z) + \epsilon^{\frac{3}{2}}W_1(R,Z) + \ldots. \end{aligned}\right\} \tag{10}$$

Substitution of (8) and (10) into (3), (4), (5) and (6) yields the following set of equations for U_0, V_0, W_0 and P_0

$$\frac{\partial P_0}{\partial R} = \frac{\partial^2 U_0}{\partial Z^2}, \quad -\frac{P_0}{R} = \frac{\partial^2 V_0}{\partial Z^2}, \quad \frac{\partial P_0}{\partial Z} = 0, \tag{11}$$

and

$$\frac{\partial U_0}{\partial R} + \frac{U_0+V_0}{R} + \frac{\partial W_0}{\partial Z} = 0; \tag{12}$$

and the following set of equations for U_1, V_1, W_1 and P_1

$$\frac{\partial P_1}{\partial R} = \frac{\partial^2 U_1}{\partial Z^2} + \frac{\partial^2 U_0}{\partial R^2} + \frac{1}{R}\frac{\partial U_0}{\partial R} - \frac{2(U_0+V_0)}{R^2}, \tag{13}$$

$$-\frac{P_1}{R} = \frac{\partial^2 V_1}{\partial Z^2} + \frac{\partial^2 V_0}{\partial R^2} + \frac{1}{R}\frac{\partial V_0}{\partial R} - \frac{2(U_0+V_0)}{R^2}, \tag{14}$$

$$\partial P_1/\partial Z = \partial^2 W_1/\partial Z^2, \tag{15}$$

and

$$\frac{\partial U_1}{\partial R} + \frac{U_1+V_1}{R} + \frac{\partial W_1}{\partial Z} = 0. \tag{16}$$

Similar sets of equations could be written down for U_i, V_i, W_i and P_i $(i > 1)$.

The boundary conditions (2) are equivalent to

$$U_i(R,O) = V_i(R,O) = W_i(R,O) = 0 \quad (i \geqslant 0) \tag{17}$$

and

$$\left.\begin{aligned} U_0(R,\delta/\epsilon)+\epsilon U_1(R,\delta/\epsilon)+\ldots &= 1,\\ V_0(R,\delta/\epsilon)+\epsilon V_1(R,\delta/\epsilon)+\ldots &= -1,\\ W_0(R,\delta/\epsilon)+\epsilon W_1(R,\delta/\epsilon)+\ldots &= 0,\end{aligned}\right\} \tag{18}$$

from which we may deduce that

$$U_0(R,H) = -V_0(R,H) = 1, \quad W_0(R,H) = 0, \tag{19}$$

and

$$8U_1(R,H) = -R^4U_{0Z}(R,H), \tag{20}$$

$$8V_1(R,H) = -R^4V_{0Z}(R,H), \tag{21}$$

$$8W_1(R,H) = -R^4W_{0Z}(R,H), \tag{22}$$

the suffix Z indicating differentiation with respect to Z. Expressions for $U_i(R,H)$, $V_i(R,H)$ and $W_i(R,H)$ $(i > 1)$ could also be written down on equating the coefficients of appropriate powers of ϵ higher than the first in equations (18).

Equations (11), (12), (17) with $i = 0$ together with (19) are of the form—as might be expected—which occur in classical lubrication theory and they are solved in the usual manner of that theory. The appropriate Reynolds equation for this set of equations is found to be

$$R^2P_0''+(R+3R^3/H)P_0'-P_0 = -6R^3/H^3. \tag{23}$$

A particular integral of (23) is

$$P_0 = 6R/5H^2, \tag{24}$$

and (24) gives the unique solution of (23) which decays to zero at $R = \infty$. This is apparent since the complementary function solutions for P_0 are $\sim R$ and $\sim R^{-1}$ for small values of R, and $\sim R^{\sqrt{10}-3}$ and $\sim R^{-(\sqrt{10}+3)}$ for large values of R. The solution $\sim R^{-1}$ for small R we exclude since P_0 must be bounded at $R = 0$, and furthermore it is impossible to construct a complementary function f which decays to zero at $R = \infty$ and is $\sim R$ for small values of R. This follows from the fact that if $f \sim +R$ for small R and f decays to zero at $R = \infty$, it possesses at least one maximum value. At the smallest value of R for which f is a maximum, $f > 0, f' = 0$ and $f'' < 0$, but from the differential equation satisfied by f, namely (23) with the right-hand side replaced by zero, $f'' > 0$ when $f' = 0$ and $f > 0$, which is a contradiction. We may similarly prove that there is no solution $f \sim -R$ for small R which decays to zero at $R = \infty$. In addition, if $P_0 \sim R^{\sqrt{10}-3}$ as $R \to \infty$, $p \sim r^{\sqrt{10}-3}/\epsilon^{\frac{1}{2}\sqrt{10}}$ when $r \gg \epsilon^{\frac{1}{2}}$. Such a property of p can only be matched with an outer solution of the kind we envisage (in which $\partial/\partial r, \partial/\partial z \sim 1$) only if $p \sim \epsilon^{-\frac{1}{2}\sqrt{10}}$ over the whole of the outer region which from physical considerations is absurd. This possibility is therefore excluded and we conclude that (24) is the only acceptable solution of (23). Below it will be shown that using (24), a completely consistent match can be obtained with the outer solution.

From (11), (12), (17), (19) and (24) it can be shown that

$$U_0 = \frac{6-9R^2}{10H^3}Z^2 + \frac{2+7R^2}{5H^2}Z, \tag{25}$$

$$V_0 = -\frac{3}{5H^2}Z^2 - \frac{2}{5H}Z, \tag{26}$$

$$W_0 = \frac{8R-2R^3}{5H^4}Z^3 + \frac{2R^3-7R}{5H^3}Z^2. \tag{27}$$

There is no difficulty, in principle, which prevents us from proceeding to calculate U_1, V_1, W_1, etc. Later in the paper (§8) we shall in fact obtain some of the salient properties of these functions but now we shall consider the contribution from the flow in the inner region to the force and couple acting on the plane and sphere.

4. Contributions to the force and couple acting on the plane and the sphere from terms of the inner solution

The contributions $(\bar{F}^i_x, \bar{F}^i_y, \bar{F}^i_z)$ to the Cartesian components of the force exerted by the fluid on the fixed plane $Z = 0$ which result from the effect of the inner solution are, in view of the boundary conditions on the plane, given by

$$\bar{F}^i_x = 6\pi\mu\mathscr{U}a\bar{f}^i, \quad \bar{F}^i_y = \bar{F}^i_z = 0,$$

where

$$6\bar{f}^i = \int_0^{R_0} \left\{\frac{\partial U}{\partial Z} - \frac{\partial V}{\partial Z}\right\}_{Z=0} R\,dR; \tag{28}$$

R_0 being a value of R chosen so that the inner solution is valid for $0 \leqslant R \leqslant R_0$. The contribution $\bar{f}^i_0$ to $\bar{f}^i$ made by the leading terms in the solution is therefore given by

$$6\bar{f}^i_0 = \int_0^{R_0} \left\{\frac{\partial U_0}{\partial Z} - \frac{\partial V_0}{\partial Z}\right\}_{Z=0} R\,dR,$$

which on evaluation using (25) and (26) gives

$$\bar{f}^i_0 = \tfrac{8}{15}\log(1+\tfrac{1}{2}R_0^2) - \tfrac{2}{5} + \tfrac{4}{5}/(2+R_0^2). \tag{29}$$

The contributions (F^i_x, F^i_y, F^i_z) to the Cartesian components of the force exerted on the sphere due to the inner solution are

$$F^i_x = -6\pi\mu\mathscr{U}af^i, \quad F^i_y = F^i_z = 0,$$

where

$$6f^i = \int_0^{\chi_0}\left\{\left[P - \epsilon^{-\frac{1}{2}}\left(2\frac{\partial U}{\partial R} - \frac{\partial V}{\partial R}\right)\right]\sin\chi + \left[\epsilon^{-1}\left(\frac{\partial U}{\partial Z} - \frac{\partial V}{\partial Z}\right) + \epsilon^{-\frac{1}{2}}\frac{\partial W}{\partial R}\right]\cos\chi\right\}_{Z=\delta/\epsilon}\sin\chi\,d\chi; \tag{30}$$

the angle $\pi - \chi$ being that between the radius vector to a surface element of the sphere from its centre and the z-axis, and χ_0 is chosen so that the inner solution is valid for $0 \leqslant \chi \leqslant \chi_0$. The contribution f^i_0 to f^i from the leading terms of the inner solution is found on using (10) and noting that $\chi = \epsilon^{\frac{1}{2}}R + O(\epsilon^{\frac{3}{2}})$ for small values of χ. The expression for f_0^i is given by

$$6f_0^i = \int_0^{R_0} \left\{RP_0 + \frac{\partial U_0}{\partial Z} - \frac{\partial V_0}{\partial Z}\right\}_{Z=H} R\,dR, \tag{31}$$

which on evaluation using (24), (25) and (26) gives

$$f_0^i = \tfrac{8}{15}\log(1+\tfrac{1}{2}R_0^2). \tag{32}$$

Expressions for the couples exerted by the fluid on the plane and the sphere may also be determined, and when moments of the surface stresses are taken about the centre of the sphere, it is found that the contributions $(\bar{G}_x^i, \bar{G}_y^i, \bar{G}_z^i)$ to the components of the couple acting on the plane which result from the inner solution are

$$\bar{G}_y^i = -8\pi\mu\mathscr{U}a^2\bar{g}^i, \quad \bar{G}_x^i = \bar{G}_z^i = 0,$$

where

$$8\bar{g}^i = -\epsilon^{\frac{3}{2}}\int_0^{R_0} P(R,0)R^2\,dR + 6(1+\epsilon)\bar{f}^i. \tag{33}$$

The contribution $\bar{g}_0^i$ to $\bar{g}^i$ due to the leading terms in the inner solution is found to be

$$\bar{g}_0^i = \tfrac{1}{10}\log(1+\tfrac{1}{2}R_0^2). \tag{34}$$

The corresponding contributions (G_x^i, G_y^i, G_z^i) to the couple acting on the sphere when moments are again taken about the centre are

$$G_y^i = 8\pi\mu\mathscr{U}a^2g^i, \quad G_x^i = G_z^i = 0,$$

where

$$8g^i = \int_0^{\chi_0}\left\{\left[\epsilon^{-1}\left(\frac{\partial U}{\partial Z} - \frac{\partial V}{\partial Z}\right) + \epsilon^{-\frac{1}{2}}\frac{\partial W}{\partial R}\right]\cos^2\chi - \left[\epsilon^{-\frac{1}{2}}\left(4\frac{\partial U}{\partial R} - \frac{\partial V}{\partial R}\right)\right]\sin\chi\cos\chi - \left[\epsilon^{-1}\frac{\partial U}{\partial Z} + \epsilon^{-\frac{1}{2}}\frac{\partial W}{\partial R}\right]\sin^2\chi\right\}_{Z=\delta/\epsilon}\sin\chi\,d\chi, \tag{35}$$

χ, χ_0 being as defined above. The contribution g_0^i to g^i which is made by the leading terms in the inner solution is seen to be

$$8g_0^i = \int_0^{R_0}\left\{\frac{\partial U_0}{\partial Z} - \frac{\partial V_0}{\partial Z}\right\}_{Z=H} R\,dR,$$

which on evaluation gives

$$g_0^i = \tfrac{1}{10}\log(1+\tfrac{1}{2}R_0^2) + \tfrac{3}{10} - \tfrac{3}{5}/(2+R_0^2). \tag{36}$$

5. The outer region

The equations governing the flow of fluid which is not in the neighbourhood of the nearest points of the sphere and the plane are again given by (1). These equations are satisfied when the pressure p and the cylindrical components (u, v, w) of $\mathbf{V}$ are of the form

$$ap = 2\mu\mathscr{U}Q\cos\theta, \quad u = \mathscr{U}[rQ + \tfrac{1}{2}(\psi + \chi)]\cos\theta, \tag{37}$$

$$v = \tfrac{1}{2}\mathscr{U}[\chi - \psi]\sin\theta, \quad w = \mathscr{U}[zQ + \phi]\cos\theta, \tag{38}$$

where r, z are cylindrical co-ordinates which are dimensionless relative to the radius of the sphere, and ψ, χ, ϕ and Q are functions of r, z only, satisfying

$$L_0^2\psi = L_2^2\chi = L_1^2\phi = L_1^2 Q = 0, \tag{39}$$

the operators being defined by (7).

The equation of continuity after substitution of u, v, w from (37) and (38) can be shown to be satisfied when

$$\left[3+r\frac{\partial}{\partial r}+z\frac{\partial}{\partial z}\right]Q+\frac{1}{2}\left[\frac{\partial\psi}{\partial r}+\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\chi+2\frac{\partial\phi}{\partial z}\right] = 0. \tag{40}$$

The boundary conditions on the plane and the sphere require that

$$\phi+zQ = 0, \quad \chi+rQ = 0, \quad \psi+rQ = 0, \qquad (41, 42, 43)$$

on the plane and that

$$\phi+zQ = 0, \quad \chi+rQ = 0, \quad \psi+rQ = 2, \qquad (44, 45, 46)$$

on the sphere.

A first approximation to the solution for the flow in the region not in the neighbourhood of the nearest points of the sphere and plane would be its limiting form when $\epsilon\to 0$ and the sphere is in contact with the plane at the origin. This limiting form is constructed by solving the problem with $\epsilon = 0$ excluding the origin from the flow region since the boundary conditions (43) and (46) cannot both be satisfied there. The fact that the component w of velocity is zero on the axis $r = 0\,(0 \leqslant z \leqslant \epsilon)$ when $\epsilon \neq 0$ and that the boundary values of w are both zero when $\epsilon = 0$ suggests that w remains bounded as $\epsilon\to 0$.

In order to facilitate the solution of the problem when $\epsilon = 0$ we transform the co-ordinates r, z to co-ordinates ξ, η by the relations

$$r = 2\eta/(\xi^2+\eta^2), \quad z = 2\xi/(\xi^2+\eta^2). \tag{47}$$

The plane is now given by $\xi = 0$, the sphere by $\xi = 1$, the origin $r = z = 0$ by $\eta = \infty$ and infinity by $\xi = \eta = 0$. It may be shown that a solution of the equation $L_m^2 f = 0$ which is non-singular at $\xi = \eta = 0$ is given by

$$f = (\xi^2+\eta^2)^{\frac{1}{2}}\int_0^\infty \{\mathscr{A}(s)\sinh s\xi+\mathscr{B}(s)\cosh s\xi\}\,J_m(s\eta)\,ds, \tag{48}$$

where $\mathscr{A}(s)$ and $\mathscr{B}(s)$ are functions of s such that the integral converges. For example, the integral converges if

$$\int_0^\infty \{\mathscr{A}(s)\sinh s+\mathscr{B}(s)\cosh s\}\,s^{-\frac{1}{2}}\,ds \text{ converges.}$$

We shall assume that there is a region of the space occupied by the fluid in which both the inner and the outer solutions are valid so that both solutions match in this region. Such an assumption implies that, in this region,

$$\epsilon^{\frac{1}{2}}R \sim 2/\eta,$$

for R and η will be large. Furthermore, in this region

$$P \sim \tfrac{24}{5}\epsilon^{-\frac{3}{2}}R^{-\frac{3}{2}} \sim \tfrac{3}{5}\eta^3.$$

Consequently, for large values of η,

$$zQ \sim \tfrac{3}{5}\xi\eta.$$

It is therefore apparent that w cannot be non-singular at the origin unless

$$\phi \sim -\tfrac{3}{5}\xi\eta \tag{49}$$

for large values of η, and in view of (39), (41) and (48) a suitable form for ϕ is given by

$$\phi = (\xi^2+\eta^2)^{\frac{1}{2}}\int_0^\infty A(s)\sinh s\xi J_1(s\eta)\,ds, \tag{50}$$

where $A(s) \sim -3/5s^2$ for small s and exponentially small for large s.

The boundary condition (41) has been satisfied in constructing ϕ given by (50) but a further relation between Q and ϕ on $\xi = 0$ may be obtained from the equation of continuity which requires $\partial w/\partial z = 0$ on the plane. On substituting for w from (38) it is seen that this relation is equivalent to

$$Q(0,\eta) = -\left.\frac{\partial\phi}{\partial z}\right|_{\xi=0} = -\lim_{\xi\to 0}\frac{\phi}{z}. \tag{51}$$

On substitution of ϕ from (50) the right-hand side of this equation becomes

$$-\tfrac{1}{2}\eta^3\int_0^\infty sAJ_1(s\eta)\,ds \sim \tfrac{3}{10}\eta^3 \tag{52}$$

for large η. It is impossible to construct a solution Q of the type (48) with $m = 1$ with a convergent integral and which $\sim 3\eta^3/10$ for large values of η. Therefore we write

$$Q = Q_0 + \tfrac{3}{10}Q_1, \tag{53}$$

where $L_1^2 Q_0 = L_1^2 Q_1 = 0$ and $Q_1(0,\eta) \sim \eta^3$ for large values of η. A suitable form for Q_1 is found to be

$$Q_1 = \eta(\xi^2+\eta^2) + (\xi^2+\eta^2)^{\frac{1}{2}}\int_0^\infty V_1(\xi,s)J_1(s\eta)\,ds, \tag{54}$$

where

$$V_1(\xi,s) = e^{-s\xi}[\xi^2/s + 3\xi/s^2] - [3e^{-s}\sinh s\xi]/s^3. \tag{55}$$

The boundary condition (51) now gives

$$\begin{aligned} Q_0(0,\eta) &= -\tfrac{3}{10}\eta^3 - \tfrac{3}{10}\eta\int_0^\infty V_1(0,s)J_1(s\eta)\,ds - \tfrac{1}{2}\eta^3\int_0^\infty sAJ_1(s\eta)\,ds \\ &= -\tfrac{1}{2}\eta^3\int_0^\infty s[A + 3/5s^2]J_1(s\eta)\,ds, \end{aligned}$$

which on integrating by parts and use of the recurrence formulae for Bessel functions may be shown to reduce to

$$Q_0(0,\eta) = \tfrac{1}{2}\eta\int_0^\infty [sA'' + A' - A/s + 9/5s^3]J_1(s\eta)\,ds,$$

the accent denoting differentiation with respect to s. The integral converges subject to the assumption we have made concerning the behaviour of A for large values of s. Consequently, a possible form for the function $Q_0(\xi, \eta)$ is given by

$$Q_0 = \tfrac{1}{2}(\xi^2+\eta^2)^{\frac{1}{2}}\int_0^\infty [B\cosh s\xi + C\sinh s\xi]\, J_1(s\eta)\, ds, \tag{56}$$

where
$$B = sA'' + A' - A/s + 9/5s^3, \tag{57}$$

and C is a function of s for which the integral is convergent. The function C is expressed in terms of A by means of the boundary condition (44) which implies that

$$\int_0^\infty [B\cosh s + C\sinh s]\, J_1(s\eta)\, ds + \frac{3}{5}\int_0^\infty V_1(1,s)\, J_1(s\eta)\, ds$$
$$= -(1+\eta^2)\left\{\int_0^\infty A\sinh s\, J_1(s\eta)\, ds + \tfrac{3}{5}\eta(1+\eta^2)^{-\frac{1}{2}}\right\}. \tag{58}$$

On using the relation

$$\eta(1+\eta^2)^{-\frac{1}{2}} = \int_0^\infty e^{-s}[1+s^{-1}]\, J_1(s\eta)\, ds, \tag{59}$$

we establish, after integration of the right-hand side of equation (58) by parts and substituting for B from (57), that (58) is satisfied when

$$C = [sA'' - A']K + \tfrac{9}{5}[e^{-s} - 1]/s^3, \tag{60}$$

where
$$K = s^{-1} - \coth s.$$

It is easy to show that $C = O(s^{-2})$ for small values of s, hence justifying our construction for Q_0 in the form given by (56).

The boundary condition (42) gives

$$\chi(0,\eta) = \eta^2\int_0^\infty sAJ_1(s\eta)\, ds. \tag{61}$$

The right-hand side $\sim -3\eta^2/5$ for large values of η, and therefore since we cannot construct a solution $\chi(\xi,\eta)$ of the form (48) with $m = 2$, with a convergent integral and which has the required asymptotic behaviour for large values of η, we write

$$\chi = \chi_0 + \tfrac{3}{5}\chi_1,$$

where $L_2^2\chi_0 = L_2^2\chi_1 = 0$ and $\chi_1(0,\eta) \sim -\eta^2$ for large η. A suitable form for χ_1 is found to be

$$\chi_1 = -\eta^2 + (\xi^2+\eta^2)^{\frac{1}{2}}\int_0^\infty V_2(\xi,s)\, J_2(s\eta)\, ds, \tag{62}$$

where
$$V_2(\xi,s) = e^{-s\xi}[\xi^2 + 3\xi/s]. \tag{63}$$

The boundary condition (61) now gives

$$\chi_0(0,\eta) = \tfrac{3}{5}\eta^2 - \tfrac{3}{5}\eta\int_0^\infty V_2(0,s)\, J_2(s\eta)\, ds + \eta^2\int_0^\infty sAJ_1(s\eta)\, ds$$
$$= \eta^2\int_0^\infty s[A + 3/5s^2]\, J_1(s\eta)\, ds,$$

which on integration by parts reduces to

$$\chi_0(0,\eta) = \eta\int_0^\infty \{A - sA' + 9/5s^2\}\, J_2(s\eta)\, ds.$$

The integral on the right-hand side of this equation is convergent. Therefore in view of (48) we may construct the function χ_0 as an integral of the form

$$\chi_0 = (\xi^2+\eta^2)^{\frac{1}{2}} \int_0^\infty [F \cosh s\xi + G \sinh s\xi] J_2(s\eta)\, ds, \tag{64}$$

where
$$F = A - sA' + 9/5s^2, \tag{65}$$

provided we can show that G is a function of s for which the integral is convergent. The function G is expressed in terms of A by means of the boundary condition (45) which implies that

$$\int_0^\infty [F \cosh s + G \sinh s] J_2(s\eta)\, ds + \frac{3}{5}\int_0^\infty V_2(1,s) J_2(s\eta)\, ds$$
$$= \eta \left\{ \int_0^\infty A \sinh s\, J_1(s\eta)\, ds + \tfrac{3}{5}[\eta/(1+\eta^2)^{\frac{1}{2}}] \right\}, \tag{66}$$

which, on integrating the right-hand side of (66) and using (65), shows that (66) is satisfied when

$$G = [2A - sA']K - 9/5s^2. \tag{67}$$

G can be shown to be $O(s^{-2})$ for small values of s, so that (64) is a proper representation for χ_0.

The determination of a suitable form for the function ψ is carried through in a similar manner, as for Q and χ. We write

$$\psi = \psi_0 + \tfrac{3}{5}\psi_1,$$

where $L_0^2 \psi_0 = L_0^2 \psi_1 = 0$ and $\psi_1(0,\eta) \sim -\eta^2$ for large η. A suitable form for ψ_1 is found to be

$$\psi_1 = -\eta^2 + (\xi^2+\eta^2)^{\frac{1}{2}} \int_0^\infty V_0(\xi, s) J_0(s\eta)\, ds, \tag{68}$$

where
$$V_0(\xi, s) = -e^{-s\xi}[\xi^2 + \xi/s] + [e^{-s} \sinh s\xi]/s^2. \tag{69}$$

The form for ψ_0 is found to be

$$\psi_0 = (\xi^2+\eta^2)^{\frac{1}{2}} \int_0^\infty [D \cosh s\xi + E \sinh s\xi] J_0(s\eta)\, ds, \tag{70}$$

where, in order to satisfy the boundary conditions (43) and (46),

$$D = sA' + A - 3/5s^2 \tag{71}$$

and
$$E = sA'K - \tfrac{3}{5}[e^{-s} - 1]/s^2 + 2(\coth s - 1). \tag{72}$$

A consideration of the asymptotic behaviour of D and E for small values of s, together with the assumed asymptotic behaviour of A for large values of s justifies the construction of ψ_0 in the form given by (70).

6. The equation of continuity

It can be easily verified that when Q, ψ, χ and ϕ satisfy the equations (39), then each of the functions

$$\left[3+r\frac{\partial}{\partial r}+z\frac{\partial}{\partial z}\right]Q, \quad \frac{\partial\psi}{\partial r}, \quad \left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\chi, \quad \frac{\partial\phi}{\partial z} \tag{73}$$

is a solution of the equation $L_1^2 f = 0$. It can further be shown that when Q, ψ, χ and ϕ are constructed in the manner described in the preceding section,

$$\begin{aligned}\left[3+r\frac{\partial}{\partial r}+z\frac{\partial}{\partial z}\right]Q &= \left[3-\xi\frac{\partial}{\partial \xi}-\eta\frac{\partial}{\partial \eta}\right]Q\\ &= \tfrac{1}{2}(\xi^2+\eta^2)^{\frac{1}{2}}\int_0^\infty \{(3B+sB')\cosh s\xi+(3C+sC'+9\,e^{-s}/5s^2)\sinh s\xi\}J_1(s\eta)\,ds,\end{aligned} \tag{74}$$

$$\begin{aligned}\frac{\partial\psi}{\partial r}+\frac{2\partial\phi}{\partial z} &= -\xi\eta\left(\frac{\partial\psi}{\partial \xi}+\frac{2\partial\phi}{\partial\eta}\right)+\tfrac{1}{2}(\xi^2-\eta^2)\left(\frac{\partial\psi}{\partial\eta}-\frac{2\partial\phi}{\partial\xi}\right)\\ &= -(\xi^2+\eta^2)^{\frac{1}{2}}\int_0^\infty\Big\{(sA''+A'-A/s+\tfrac{1}{2}sD''+9/5s^3)\cosh s\xi\\ &\quad+\left(\tfrac{1}{2}sE''-9/5s^3+\tfrac{3}{5}e^{-s}\left[\frac{3}{s^3}+\frac{2}{s^2}+\frac{1}{2s}\right]\right)\sinh s\xi\Big\}J_1(s\eta)\,ds,\end{aligned} \tag{75}$$

$$\begin{aligned}\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\chi &= -\xi\eta\frac{\partial\chi}{\partial\xi}+\tfrac{1}{2}(\xi^2-\eta^2)\frac{\partial\psi}{\partial\eta}+\frac{(\xi^2+\eta^2)}{\eta}\chi\\ &= \tfrac{1}{2}(\xi^2+\eta^2)^{\frac{1}{2}}\int_0^\infty\{(sF''+4F'+2F/s)\cosh s\xi\\ &\quad+(sG''+4G'+2G/s)\sinh s\xi\}J_1(s\eta)\,ds.\end{aligned} \tag{76}$$

From (74) to (76) we observe that the equation of continuity (40) is of the form

$$(\xi^2+\eta^2)^{\frac{1}{2}}\int_0^\infty \{\lambda(s)\cosh s\xi+\mu(s)\sinh s\xi\}J_1(s\eta)\,ds = 0, \tag{77}$$

where $\lambda(s)$ involves only the functions A, B, D and F and $\mu(s)$ involves only the functions C, E and G. If B, D and F are expressed in terms of A by the relations (57), (65) and (71) we find that $\lambda(s) \equiv 0$; this of course must be so since the equation of continuity when $\xi = 0$ has already been used in the derivation of the relations. For the equation of continuity to be satisfied at all points of the fluid with the exception of the origin, it therefore follows that $\mu(s) = 0$. Consequently on expressing C, E and G in terms of A by the relations (60), (67) and (72) it follows that

$$s^3K'A''+sA'[s^2K''+3sK'+2K]-A[s^2K''+4sK'+2K]+s^2X'' = 0, \tag{78}$$

where $K = s^{-1}-\coth s$ and $X = \coth s-1$. The solution to (78) which we seek must be $\sim -3/5s^2$ for small values of s and must decay to zero exponentially for large values of s. It has not been possible to obtain this solution in closed form.

It may be easily shown that if A_I and A_{II} are the particular integral and complementary function respectively of (78) for small values of s,

$$A_I = -3/5s^2 + O(1),$$
$$A_{II} = s^{\sqrt{10}-2}\{1+O(s^2)\};$$

the other complementary function solution which $\sim s^{-(\sqrt{10}+2)}$ for small values of s we reject in view of the prescribed asymptotic behaviour of the required solution for small s.

It may further be shown that if A_1, A_2 and A_3 are the particular integral and complementary function solutions respectively of (78) for large values of s, then

$$A_1 = -2s^2e^{-2s} - O(s^4e^{-4s}), \quad A_2 = 1-2s+O(se^{-2s}), \quad A_3 = e^{-2s}+O(s^2e^{-4s}).$$

In view of the singular behaviour of A_I, A_I', A_I'' and A_{II}'' at $s = 0$, in order to perform the numerical integration of (78) it is necessary to have accurate expressions for A_I and A_{II}; if one aims at obtaining A correct to five decimal places say, it is essential to know the expansion of A_I as far as the term in s^4 and A_{II} as far as the term in $s^{\sqrt{10}+4}$ at least. The equation (78) was integrated numerically twice for values of s up to 5·6; the first time using the expansion of A_I for the starting values and the second time using the expansion of A_{II} for the starting values and with X'' set at zero. The solution for the present problem was then found by choosing that linear combination of the two numerical solutions which

$$\sim -2s^2e^{-2s} + O(e^{-2s})$$

for large values of s.

7. Contributions to the force and couple acting on the plane and sphere due to the outer solution

The contributions $(\bar{F}^0_x, \bar{F}^0_y, \bar{F}^0_z)$ to the Cartesian components of the force acting on that part of the fixed plane $\xi = 0$ for which $0 \leqslant \eta \leqslant \eta_0$ which result from the outer solution are given by

$$\bar{F}^0_x = 6\pi\mu\mathscr{U}a\bar{f}^0, \quad \bar{F}^0_y = \bar{F}^0_z = 0,$$

where
$$3\bar{f}^0 = \int_0^{\eta_0}\left\{\frac{2}{\eta^2}\frac{\partial Q}{\partial\xi} + \frac{1}{\eta}\frac{\partial\psi}{\partial\xi}\right\}_{\xi=0} d\eta. \tag{79}$$

The contribution $\bar{f}^0_0$ to $\bar{f}^0$ from the solution which we have constructed in the previous section is therefore given by

$$3\bar{f}^0_0 = \int_0^{\eta_0}\left\{\int_0^\infty [s^2A''-sA']KJ_1(s\eta)\,ds\right\}\eta^{-1}d\eta + \int_0^{\eta_0}\left\{\int_0^\infty [s^2A'K+2s(\coth s-1)]J_0(s\eta)\,ds\right\}d\eta; \tag{80}$$

K is as defined for (78). Consideration of A and its derivatives for small values of s shows that

$$[s^2A''-sA'] = 8/5s + O(s^3),$$
$$[s^2A'K + 2s(\coth s - 1)] = 8/5 + O(s^2).$$

It is therefore clear that if the order of integration in the double integrals of (80) is reversed and η_0 allowed to approach infinity, the integrals will no longer converge. However if we write

$$3\bar{f}_0^0 = \int_0^{\eta_0}\left\{\int_0^\infty ([s^2A' - sA']K - 8e^{-2s}/5s)\,J_1(s\eta)\,ds\right\}\eta^{-1}d\eta$$
$$+\int_0^{\eta_0}\left\{\int_0^\infty (s^2A'K + 2s\,(\coth s - 1) - 8e^{-2s}/5)\,J_0(s\eta)\,ds\right\}d\eta$$
$$+\frac{8}{5}\int_0^{\eta_0}\int_0^\infty e^{-2s}s^{-1}J_1(s\eta)\,ds\,\eta^{-1}d\eta + \frac{8}{5}\int_0^{\eta_0}\int_0^\infty e^{-2s}J_0(s\eta)\,ds\,d\eta, \qquad (81)$$

the order of integration in the first two of the double integrals of (81) may now be reversed when $\eta_0 \to \infty$. Consequently

$$\bar{f}_0^0 = \frac{1}{3}\int_0^\infty \{s^2A''K + 2\,(\coth s - 1) - 16e^{-2s}/5s\}\,ds$$
$$+\tfrac{16}{15}\log\eta_0 - \tfrac{8}{15} + p_1(\eta_0), \qquad (82)$$

where $p_1(\eta_0) \to 0$ as $\eta_0 \to \infty$. The total force acting on the plane may now be obtained by combining the contributions due to both the inner and outer solutions as given by (29) and (82), provided that we identify the values of r corresponding to η_0 and R_0 and set

$$R_0\eta_0 = 2/\epsilon^{\frac{1}{2}}. \qquad (82)$$

It is tacitly assumed here that the inner and outer solutions have a region of overlap in which $1 \ll R \ll \epsilon^{-\frac{1}{2}}$ and $1 \ll \eta \ll \epsilon^{-\frac{1}{2}}$. The first-order approximation, neglecting those terms which tend to zero with ϵ, then gives

$$\bar{F}_x = 6\pi\mu\mathscr{U}a\bar{f}_0, \quad \bar{F}_y = \bar{F}_z = 0,$$

where $\bar{f}_0 = \bar{f}_0^i + \bar{f}_0^0$. On using (29), (82) and (83), we obtain

$$\bar{f}_0 = \tfrac{8}{15}\log(2/\epsilon) - \tfrac{14}{15} + \tfrac{1}{3}\int_0^\infty \{s^2A''K + 2\,(\coth s - 1) - 16e^{-2s}/5s\}\,ds + p_1(\eta_0) + p_2(R_0),$$

where $p_1(\eta_0) \to 0$ as $\eta_0 \to \infty$ and $p_2(R_0) \to 0$ as $R_0 \to \infty$. Hence proceeding to these limits, we have

$$\bar{f}_0 = \tfrac{8}{15}\log(2/\epsilon) - \tfrac{14}{15} + \frac{1}{3}\int_0^\infty \{s^2A''K + 2\,(\coth s - 1) - 16e^{-2s}/5s\}\,ds. \qquad (84)$$

The discussion of the force on the sphere and of the couples acting on both the sphere and the plane follows similar lines. We shall therefore not give the details but content ourselves with quoting the principal results. The contributions (F_x^0, F_y^0, F_z^0) to the Cartesian components of the force acting on the sphere $\xi = 1$ due to the outer solution are

$$F_x^0 = -6\pi\mu\mathscr{U}af^0, \quad F_y^0 = F_z^0 = 0,$$

where
$$f^0 = \frac{1}{3}\int_0^{\eta_0}\left\{r\frac{\partial Q}{\partial\xi} - Q\frac{\partial r}{\partial\xi} + \frac{\partial\psi}{\partial\xi}\right\}_{\xi=1}\frac{\eta\,d\eta}{1+\eta^2}; \qquad (85)$$

η_0 being defined as before. The leading term f_0^0 of f^0 is given by

$$f_0^0 = \tfrac{16}{15}\log\eta_0 - \tfrac{14}{15} + \frac{1}{3}\int_0^\infty \{s^2A''K + 2(\coth s - 1) - 16e^{-2s}/5s\}\,ds + q_1(\eta_0), \tag{86}$$

where $q_1(\eta_0) \to 0$ as $\eta_0 \to \infty$.

Combining (86) with (32) we obtain the total force exerted on the sphere by the fluid to a first approximation. This is given by

$$F_x = 6\pi\mu\mathscr{U}af_0, \quad F_y = F_z = 0, \tag{87}$$

where $$f_0 = \tfrac{8}{15}\log(2/\epsilon) - \tfrac{14}{15} + \frac{1}{3}\int_0^\infty \{s^2A''K + 2(\coth s - 1) - 16e^{-2s}/5s\}\,ds.$$

It is noted that $f_0 = \bar{f}_0$ so that the forces which act on the sphere and plane are equal and opposite in agreement with a more general result of O'Neill (1964*a*).

On taking moments about the centre of the sphere, the Cartesian components $(\bar{G}_x^0, \bar{G}_y^0, \bar{G}_z^0)$ of the couple acting on the plane due to the fluid motion and resulting from the outer solution are given by

$$\bar{G}_x^0 = 0, \quad \bar{G}_y^0 = -8\pi\mu\mathscr{U}a^2\bar{g}^0, \quad \bar{G}_z^0 = 0, \tag{88}$$

where $$\bar{g}^0 = \int_0^{\eta_0}\left\{\frac{\partial\phi}{\partial\xi}\right\}_{\xi=0}\frac{d\eta}{\eta^2} + \tfrac{3}{4}\bar{f}_0.$$

The leading term $\bar{g}_0^0$ of $\bar{g}^0$ is

$$\bar{g}_0^0 = \int_0^\infty \{sA + 3e^{-2s}/5s\}\,ds + \tfrac{2}{5}(1 - \log\eta_0) + \tfrac{3}{4}\bar{f}_0^0 + q_2(\eta_0), \tag{89}$$

where $q_2(\eta_0) \to 0$ as $\eta_0 \to \infty$. Combining (89) with (34), we find that the leading term contribution to the couple on the plane from both the inner and the outer solutions has Cartesian components

$$\bar{G}_x = 0, \quad \bar{G}_y = -8\pi\mu\mathscr{U}a^2\bar{g}_0, \quad \bar{G}_z = 0, \tag{90}$$

where $$\bar{g}_0 = \tfrac{1}{5} + \tfrac{1}{10}\log(2/\epsilon) + \frac{1}{4}\int_0^\infty \{4sA + s^2A''K + 2(\coth s - 1) - 4e^{-2s}/5s\}\,ds.$$

Finally, again taking moments about the centre of the sphere the leading term in the couple exerted by the fluid on the sphere due to the combined effects of the inner and outer solutions has Cartesian components

$$G_x = 0, \quad G_y = 8\pi\mu\mathscr{U}a^2g_0, \quad G_z = 0, \tag{91}$$

where

$$\begin{aligned} g_0 = {} & \tfrac{4}{5} + \tfrac{1}{10}\log(2/\epsilon) \\ & + \frac{1}{2}\int_0^\infty \left\{\left(A + \frac{3}{5s^2}\right)[2s\,\mathrm{cosech}^2 s - (\coth s - 1)(1 + 2s + s^2\,\mathrm{cosech}^2 s)]\right. \\ & \left. - \tfrac{3}{5}e^{-2s}\left[\frac{1}{s^3} + \frac{1}{s^2} + \frac{2}{3s} + \tfrac{5}{3}(2s-1)\coth s\right] - \frac{3}{5}\left[\frac{2}{s^3} - \left(\frac{3}{s^2} + \frac{2}{s}\right)(\coth s - 1)\right]\right\} ds. \end{aligned}$$

8. The contributions $O(\epsilon \log \epsilon)$ to the forces and couples

In deriving the formulae (84), (87), (90) and (91) for the forces and couples acting on the sphere and the plane, we neglected terms $O(\epsilon)$. It is evident also that if we now proceeded to determine the next term in the inner solution expansions (10) of the velocities and pressure and the corresponding outer solution, we would be able to obtain correction terms up to $O(\epsilon)$ to the expressions for the forces and couples. As with the leading terms, it is to be expected that terms $O(\epsilon \log \epsilon)$ will appear in conjunction with terms $O(\epsilon)$. In this section we shall determine the terms $O(\epsilon \log \epsilon)$ explicitly for which it is only necessary to examine the structure of U_1 and V_1 as $R \to \infty$, whereas to determine the terms $O(\epsilon)$ we need to know the structure of U_1 and V_1 for *all* R.

The reason is as follows: from (28) it is clear that the contribution $\bar{f}_1^i$ to $\bar{f}^i$, the reduced force on the plane, from the second-order solution is

$$\bar{f}_1^i = \frac{\epsilon}{6} \int_0^{R_0} \left\{ \frac{\partial U_1}{\partial Z} - \frac{\partial V_1}{\partial Z} \right\}_{Z=0} R\, dR.$$

Now it turns out, in a manner to be demonstrated below, that when R is large,

$$R\{(\partial U_1/\partial Z) - (\partial V_1/\partial Z)\}_{Z=0} = a_{11} R + (a_{12}/R) + O(R^{-3}). \tag{92}$$

Hence when R_0 is large,

$$\bar{f}_1^i = \tfrac{1}{6}\epsilon\{\tfrac{1}{2}a_{11} R_0^2 + a_{12} \log R_0 + A_1^i + O(R_0^{-2})\},$$

where A_1^i is a constant which depends on the overall properties of U_1 and V_1. If we now relate R_0 to the corresponding variable $r_0 = 2\eta_0/(1+\eta_0^2)$ of the outer solution by means of the formula $r_0 = \epsilon^{\frac{1}{2}} R_0$, it follows that

$$\bar{f}_1^i = \tfrac{1}{6}\{\tfrac{1}{2}a_{11} r_0^2 + \tfrac{1}{2}a_{12}\epsilon \log \epsilon + a_{12}\epsilon \log r_0 + \epsilon A_1^i + O(\epsilon/r_0^2)\}, \tag{93}$$

where R_0 is large but r_0 is small. Now let us consider the outer solution. On computing the force on the outer part of the plane for which $r \geqslant r_0$ and adding to (93), we can obtain the total force on the plane to order ϵ. Such an expression must, however, be independent of r_0 for the choice of r_0 is to some extent arbitrary and hence the terms which depend on r_0 must cancel when (93) is added to the contribution from the outer solution. Of these terms, the first, $\frac{1}{12}a_{11} r_0^2$, being independent of ϵ must be cancelled by part of $\bar{f}_0^0$. It is inferred that $\bar{f}_0^0$ contains a term $-\frac{1}{12}a_{11} r_0^2$. The term $\frac{1}{6}a_{12}\epsilon \log r_0$ is of order ϵ and so must be cancelled by the principal part of $\bar{f}_1^0$. It is inferred that

$$\bar{f}_1^0 = -\tfrac{1}{6}\epsilon a_{12} \log r_0 + \tfrac{1}{6}\epsilon A_1^0 + O(1),$$

where r_0 is small and where A_1^0 is a constant depending on the overall properties of the terms $O(\epsilon)$ in the outer solution. Hence

$$\bar{f}_1 = \tfrac{1}{12}a_{12}\epsilon \log \epsilon + \tfrac{1}{6}\epsilon(A_1^i + A_1^0).$$

Similar remarks apply to the force on the sphere and to the couples. As we have already remarked, in order to determine a_{12} it is only necessary to examine the limiting structure of U_1 and V_1 as $R \to \infty$. This we do in the following way.

From equation (15)

$$P_1(R,Z) = (\partial W_0/\partial Z)+F(R), \tag{94}$$

which together with (25), (26) and (27) on substitution into (13) and (14) gives, for large values of R,

$$\frac{\partial^2 U_1}{\partial Z^2} = F'(R)+\left[\frac{192}{R^6}-\frac{16{,}128}{5}\frac{1}{R^8}+O\left(\frac{1}{R^{10}}\right)\right]Z^2+\left[-\frac{32}{R^4}+\frac{3072}{5}\frac{1}{R^6}+O\left(\frac{1}{R^8}\right)\right]Z, \tag{95}$$

$$\frac{\partial^2 V_1}{\partial Z^2} = -\frac{F(R)}{R}+\left[\frac{192}{R^6}-\frac{2304}{5}\frac{1}{R^8}+O\left(\frac{1}{R^{10}}\right)\right]Z^2+\left[\frac{32}{5}\frac{1}{R^4}+O\left(\frac{1}{R^8}\right)\right]Z. \tag{96}$$

The boundary conditions which must be satisfied by U_1, V_1 and W_1 are, from (17) with $i = 1$, (20), (21) and (22),

$$U_1(R,0) = V_1(R,0) = W_1(R,0) = 0,$$

and for large values of R,

$$U_1(R,H) = [\tfrac{2}{5}-(4/R^2)]\,H+O(1/R^2),$$
$$V_1(R,H) = [\tfrac{4}{5}-(16/5R^2)]\,H+O(1/R^2),$$
$$W_1(R,H) = [(8/5R^3)-(112/5R^5)]\,H^3+O(1/R). \tag{97}$$

After integrating (95) and (96) twice with respect to Z and using (17) and (97), one obtains

$$U_1(R,Z) = \tfrac{1}{2}F'(R)(Z^2-ZH)+\left[\frac{16}{R^6}-\frac{1344}{5R^8}+O\left(\frac{1}{R^{10}}\right)\right](Z^4-ZH^3)$$
$$+\left[-\frac{16}{3R^4}+\frac{512}{5R^6}+O\left(\frac{1}{R^8}\right)\right](Z^3-ZH^2)+\left[\frac{2}{5}-\frac{4}{R^2}+O\left(\frac{1}{R^4}\right)\right]Z, \tag{98}$$

$$V_1(R,Z) = -\frac{1}{2}\frac{F(R)}{R}(Z^2-ZH)+\left[\frac{16}{5R^6}-\frac{192}{5R^8}+O\left(\frac{1}{R^{10}}\right)\right](Z^4-ZH^3)$$
$$+\left[\frac{16}{15R^4}+O\left(\frac{1}{R^8}\right)\right](Z^3-ZH^2)+\left[\frac{4}{5}-\frac{16}{5R^2}+O\left(\frac{1}{R^4}\right)\right]Z, \tag{99}$$

for large values of R and from (16),

$$W_1(R,Z) = -\int_0^Z\left\{\frac{\partial U_1}{\partial R}+\frac{(U_1+V_1)}{R}\right\}dZ. \tag{100}$$

On putting $Z = H$ in equation (100) after substituting U_1 and V_1 from (98) and (99), the following second-order differential equation for $F(R)$ is obtained

$$R^2F''+\left(R+\frac{3R^3}{H}\right)F'-F = -\frac{72}{25}\frac{1}{R}-\frac{256}{5}\frac{1}{R^3}+O\left(\frac{1}{R^5}\right). \tag{101}$$

The left-hand side of equation (101) is exact; it therefore follows that the complementary function solutions of the differential equation which is satisfied by F for arbitrary values of R are the complementary function solutions of (23). Therefore by the same reasoning which we employed to establish that (24) was *the* solution of (23) for P_0, we assert that the complementary function solutions of (101) may be rejected and consequently that, for large values of R,

$$F(R) = \frac{12}{15}\frac{1}{R}+\frac{712}{125}\frac{1}{R^3}+O\left(\frac{1}{R^5}\right). \tag{102}$$

Hence on substituting from (102) into (98) and (99) and then into (92),

$$a_{12} = \tfrac{256}{125},$$

and we conclude that

$$\bar{f} = \bar{f}^i + \bar{f}^0 = \bar{f}_0 - \tfrac{64}{375}\epsilon\log\epsilon + O(\epsilon). \tag{103}$$

In a similar way it may be shown that the reduced couple on the plane is

$$\bar{g} = \bar{g}^i + \bar{g}^0 = \bar{g}_0 - \tfrac{43}{250}\epsilon\log\epsilon + O(\epsilon), \tag{104}$$

the reduced force on the sphere is

$$f = f^i + f^0 = f_0 - \tfrac{64}{375}\epsilon\log\epsilon + O(\epsilon), \tag{105}$$

and the reduced couple on the sphere is

$$g = g^i + g^0 = g_0 - \tfrac{43}{250}\epsilon\log\epsilon + O(\epsilon). \tag{106}$$

9. Results of the numerical work.

The numerical values of the solution $A(s)$ of the differential equation (78) which $\sim -3/5s^2 + O(1)$ for small values of s and $\sim -2s^2e^{-2s} + O(e^{-2s})$ for large values of s were tabulated to an accuracy of five decimal places in the range

$$0{\cdot}05(0{\cdot}05)2{\cdot}0(0{\cdot}2)5{\cdot}6.$$

The integrals which appear in the expressions for $\bar{f}_0$, f_0, $\bar{g}_0$ and g_0 were then evaluated, as a result of which, we may write (103)–(106) as

$$\left.\begin{aligned} \bar{f} = f &= [\tfrac{8}{15} + \tfrac{64}{375}\epsilon]\log(2/\epsilon) + 0{\cdot}58461 + O(\epsilon), \\ \bar{g} &= [\tfrac{1}{10} + \tfrac{43}{250}\epsilon]\log(2/\epsilon) - 0{\cdot}26221 + O(\epsilon), \\ g &= [\tfrac{1}{10} + \tfrac{43}{250}\epsilon]\log(2/\epsilon) - 0{\cdot}26227 + O(\epsilon). \end{aligned}\right\} \tag{107}$$

The values of $\bar{f}$, f, $\bar{g}$ and g, ignoring terms of $O(\epsilon)$ and higher, are given in table 1 for a range of small values of ϵ and the corresponding values of f and g which were obtained by O'Neill (1964*a*) which in the notation of that paper are written as F^* and G^* respectively, are given here for comparison. A satisfyingly close agreement is found between the results of the approximate theory which we have developed in this paper and those predicted by O'Neill's exact theory.

ϵ	f	F^*	$\bar{g}$	g	G^*
0·00020	5·4971	5·4973	0·65914	0·65908	0·65912
0·00045	5·0649	5·0651	0·57837	0·57831	0·57834
0·00080	4·7584	4·7587	0·52126	0·52120	0·52120
0·00125	4·5208	4·5123	0·47713	0·47707	0·47706
0·00180	4·3269	4·3275	0·44124	0·44118	0·44116
0·00245	4·1631	4·1639	0·41106	0·41100	0·41096
0·00320	4·0213	4·0223	0·38506	0·38500	0·38496
0·00405	3·8964	3·8976	0·36226	0·36220	0·36214
0·00500	3·7847	3·7863	0·34201	0·34195	0·34187

TABLE 1

An interesting feature of these results is the almost identical values which are given for $\bar{g}$ and g, which leads one to conjecture whether or not the couples acting on the sphere and plane as well as the forces are equal and opposite. However it may be shown either by considering the asymptotic form of the solution given by O'Neill (1964*a*) for large values of the parameter α of that paper or by the methods of reflexions, that for the problem when $\epsilon \gg 1$,

$$g \sim 3\epsilon^{-4}/32, \quad \bar{g} \sim 5\epsilon^{-4}/64.$$

The expressions which we have obtained for $\bar{g}$ and g suggest that they are of the form

$$\left[\sum_{n=0}^{\infty} \alpha_n \epsilon^n\right] \log \epsilon + \sum_{n=0}^{\infty} \beta_n \epsilon^n.$$

It is possible that the corresponding coefficients α_n are the same for both series but the corresponding coefficients β_n are different; this together with the almost identical coefficients β_0 in each series would give very similar values for $\bar{g}$ and g for the very small values of ϵ when the terms $O(\log \epsilon)$, $O(\epsilon \log \epsilon)$ and $O(1)$ dominate. It is further possible that β_0 is the same for both expansions, a fact we have not been able to establish.

10. The theory of lubrication

Although the theory of lubrication has been of great interest for many years and has been applied to a large number of separate flow problems, little effort has been put into establishing its validity either from an experimental or a theoretical standpoint. In their discussion of the available experiments on bearings, Pinkus & Sternlicht (1961) point out that while there is an abundance of test data, significant experiments on bearings are rare.

One reason lies in the difficulty of constructing and maintaining bearings with minute clearances of the necessary accuracy. Another more important reason concerns the lubricant viscosity. To quote Pinkus & Sternlicht (p. 426): 'It is almost impossible to vary a parameter during testing without simultaneously varying the viscosity field of the lubricant. The only possible escape from this difficulty would be to use a fluid whose viscosity is not affected by temperature, pressure or rate of shear. Such a lubricant, however, does not exist'. The experiments they report, however, show an encouraging agreement with theory, in that part of the flow region where it may be expected to hold, provided neither large subatmospheric loops in the pressure nor cavitation occur. From a theoretical standpoint too, little has hitherto been achieved towards embedding the theory in a theory of hydrodynamics based on the Navier–Stokes equations and in assessing the errors made in applying the theory to specific problems. Reference may be made to Langlois (1964) and Thompson (1964) for a discussion of the present state of these aspects of the theory.

In view of these remarks, it is believed that the flow discussed in this paper is of some use for assessing the merits of lubrication theory, for it has in some sense one of the very few known exact solutions of the Navier–Stokes equations, in which the flow region contains a region to which lubrication theory can be applied. Furthermore, the flow is realizable in a laboratory, particularly if use is made of the natural extension of this work to the case when the sphere rotates which has been carried out recently by M. D. Cooley and one of us (M.E. O'N.) and which it is hoped to publish elsewhere.

We find that in the neighbourhood of O, the nearest point of the plane to the sphere, lubrication theory gives a description of the local flow properties accurately to $O(\epsilon)$ where ϵa is the minimum clearance between the sphere and the plane, as indeed would be expected. This theory is, however, of more limited validity when it is used to compute the overall forces or couples, because it is then only on an equal footing with the theory of weakly-sheared flow past the rest of the sphere. The force and couple on the sphere, for instance, predicted by lubrication theory, are of the form $A \log \epsilon + B$, where A and B are independent of ϵ if terms of $O(\epsilon)$ are neglected, and the value of B is uncertain since it depends on the limits assigned to the lubrication zone. The actual force and couple are $A \log \epsilon + C + O(\epsilon \log \epsilon)$, where C has been determined explicitly (107) and depends substantially on the flow past the rest of the sphere.

We conclude therefore that while lubrication theory accurately describes *local* flow properties, it does not provide a reliable estimate of *overall* flow properties, particularly forces and couples, if the flow region includes a substantial region of weakly sheared flow. Nevertheless, its predictions of the forces and couples in the present problem are formally correct in the limit $\epsilon \to 0$.

REFERENCES

BURGERS, J. M. 1941 *Proc. Koningl. Akad. Wetenschap (Amsterdam)*, **44**, 1054.

DEAN, W. R. & O'NEILL, M. E. 1963 *Mathematika*, **10**, 13.

FAXÉN, H. Z. 1927 *Angew. Math. Mech.* **7**, 79.

GROSS, W. A. 1962 *Gas Film Lubrication*. New York: Wiley.

HAPPEL, J. & BRENNER, H. 1965 *Low Reynolds Number Hydrodynamics*. New York: Prentice-Hall.

KYNCH, G. J. 1959 *J. Fluid Mech.* **5**, 193.

LANGLOIS, W. E. 1964 *Slow Viscous Flow*. New York: Macmillan.

MAJUMDAR, S. R. 1965 University of London Ph.D. Thesis.

O'NEILL, M. E. 1964*a* *Mathematika*, **11**, 67.

O'NEILL, M. E. 1964*b* University of London Ph.D. Thesis.

PINKUS, O. & STERNLICHT, B. 1961 *Lubrication Theory*. New York: McGraw-Hill.

SMOLUCHOWSKI, M. 1911 *Bull. Inter. acad. Polonaise Sci. lett.* **1** A, 28.

STIMSON, M. & JEFFREY, G. B. 1926 *Proc. Roy. Soc.* A, **111**, 110.

THOMPSON, B. W. 1964 M.Sc. Thesis, Melbourne.